Overview of Logic and Computation: Notes

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1 To begin at the beginning...

We study formal logic as a mathematical tool for reasoning and as a medium for knowledge representation. The central notion is that of a *consequence relation* defined over a formal language of some kind. This is intended to capture the intuitive concept of valid inference or entailment, at least in that it provides a catalogue of valid argument *forms*.

1.1 Consequence relations

At the most abstract level, we do not say what is in the language, except that it is a set of objects which we may call sentences or, more neutrally, formulae. A set of formulae (assumptions or premises) entails a single formula (conclusion) if there is no way all of the premises could hold without the conclusion holding as well. The relation of entailment, which we shall symbolise with a "turnstile" ($\vdash$), is a consequence relation in the technical sense if and only if it satisfies three conditions. For all sets Γ, Δ and for all formulae A, B:

1. If $A \in \Gamma$ then $\Gamma \vdash A$.
2. If $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash A$.
3. If $\Gamma \vdash A$ for every A in Δ, and $\Delta \vdash B$, then $\Gamma \vdash B$.

Condition 1 is an identity postulate: in asking whether a conclusion follows from assumptions, or whether a query succeeds from a database, the answer is yes if the conclusion is one of the assumptions, or if the query is already explicitly in the database. Condition 2 imposes a requirement of monotonicity, also known as *weakening*: for the query to succeed, it is OK to use only as much of the database as is needed. Condition 3, called *cut* in the literature of logic, is a kind of transitivity: if the lemmas follow from the axioms and the theorem from the lemmas, that's enough for a proof of the theorem from the axioms.

1.2 First order language

To make things a little more concrete, we now consider the standard first order logic or [*lower*] *predicate calculus*. For this, we suppose we have denumerably many *variables* $x_1 \ldots x_n \ldots$ and also *function symbols* f_i^n and *predicate symbols* P_i^n for every *arity* $n \geq 0$. As usual, we call nullary functions *constants*, and nullary predicate symbols *propositional symbols*. We also have one special binary predicate '=' called *identity*, whose specialness will only matter when we define *models*. A *term* is either a variable or an n-ary function symbol followed by n terms. An *atomic formula* is an n-ary predicate symbol followed by n terms. To these we apply the *connectives* $\neg$, $\wedge$, $\vee$ and $\rightarrow$ and *quantifiers* $\forall x_i$ and $\exists x_i$ binding variable x_i. A variable is free in a formula if it is not inside the scope of any quantifier binding it. Term t is free for variable x in formula A iff x does not occur free in A inside the scope of a quantifier binding a variable which occurs in t. The *propositional constants* $\top$ and $\bot$ (true and false) are available as nullary connectives (taking no arguments and returning a sentence).

We allow ourselves some freedom of formal vocabulary, dropping subscripts and superscripts as much as is reasonable, writing certain binary function and predicate symbols between arguments and not in front of them, using 'y' and 'z' as ways of spelling variables such as x_i and x_j, and applying parentheses in the obvious way. Thus we can get away with writing $x + (y \cdot z) = t$ rather than $=^2 +^2 x_0 \cdot^2 x_0 x_{13} x_5$.

We use upper case A, B, etc as metalinguistic variables over formulae and Γ, Δ, etc to stand for sets of formulae. We write, for example, Γ, A, B instead of $\Gamma \cup \{A, B\}$ in order to reduce clutter. When we want to indicate that a sequent is provable in some proof system we write it with a single 'turnstile' $\vdash$, subscripting with the name of the system if ambiguity is possible. To say that the sequent is valid in some semantics we sometimes use a double turnstile $\models$ in place of the single one.

1.3 Interpretations, models and satisfaction

An interpretation $\mathcal{I}$ of the propositional fragment (with only nullary predicate symbols and no terms or quantifiers) is a function assigning to each propositional symbol one of two values. These might as well be 0 and 1, so let us fix them as such. $\mathcal{I}$ induces a notion of truth: P is true for $\mathcal{I}$ iff $\mathcal{I}(P) = 1$. $\neg A$ is true for $\mathcal{I}$ iff A is false (not true) for $\mathcal{I}$. $A \wedge B$ is true for $\mathcal{I}$ iff both conjuncts are, and similarly for the other connectives according to their truth tables. $\mathcal{I}$ is a model of Γ iff A is true for $\mathcal{I}$ for all $A \in \Gamma$. $\Gamma \models A$ iff A is true for every model of Γ. More generally (though we shall not make much of this generalisation) we may say that for sets Γ and Δ, $\Gamma \models \Delta$ iff for every model of Γ at least one formula in Δ is true.

Interpretations of the full first order language have to be more elaborate because the language is so much richer. An interpretation consists of a nonempty set $\mathcal{D}$ called the *domain* or *universe* and an interpretation function $\mathcal{I}$ which assigns to each f^n a total function from $\mathcal{D}^n$ into $\mathcal{D}$ and to each predicate symbol P^n an n-ary relation over $\mathcal{D}$ (that is, a subset of $\mathcal{D}^n$).[1] The identity predicate is always interpreted as the diagonal relation over $\mathcal{D}$—that is, the relation that every object bears to itself and to nothing else. A valuation over domain $\mathcal{D}$ is a function v assigning to each variable x a member of $\mathcal{D}$. In terms of this, we can define denotation δ. For variable x, $\delta(x) = v(x)$. For compound terms, $\delta(ft_1 \ldots t_n) = \mathcal{I}(f)(\delta(t_1) \ldots \delta(t_n))$. We say that an interpretation and valuation satisfies atomic $Pt_1 \ldots t_n$ iff $\langle \delta(t_1) \ldots \delta(t_n) \rangle \in \mathcal{I}(P)$. Satisfaction extends in the

[1]First order interpretations extend propositional ones because $\mathcal{D}^0$ is the set of all functions from $\emptyset$ to $\mathcal{D}$ and the empty function is a function (that is, empty set satisfies set-theoretical definition of function); therefore $\mathcal{D}^0 = \{\emptyset\}$ and this has exactly two subsets: $\emptyset$ and $\{\emptyset\}$, that is 0 and 1.

obvious way to compounds built by applying connectives: e.g. an interpretation satisfies $A \to B$ iff either it satisfies B or it does not satisfy A. For quantifiers we need the notion of an x-variant of valuation v, which is simply a valuation w such that for all y other than x, $v(y) = w(y)$. Then $\forall x A$ is satisfied by v iff A is satisfied by all x-variants of v, and similarly $\exists x A$ is satisfied by v if A is satisfied by some x-variant of v. A is true for $\mathcal{I}$ iff it is satisfied by all valuations under $\mathcal{I}$. $\mathcal{I}$ is a model of Γ iff every formula in Γ is true for $\mathcal{I}$. $\Gamma \models A$ iff every valuation under every interpretation which satisfies every formula in Γ satisfies A. Again, this definition generalises to the multiple conclusion case: $\Gamma \models \Delta$ iff every valuation under every interpretation which satisfies every formula in Γ satisfies at least one formula in Δ.

1.4 Exercise

Show that the following are true:

1. $\models$ is a consequence relation.
2. $\Gamma \models A \wedge B$ iff $\Gamma \models A$ and $\Gamma \models B$.
3. $\Gamma \models \neg A$ iff $\Gamma, A \models \bot$.
4. $\Gamma \models A \to B$ iff $\Gamma, A \models B$.
5. If $\Gamma \models \bot$ then $\Gamma \models A$ for every formula A.
6. If A contains no free variables and A is satisfied by a valuation v under interpretation $\mathcal{I}$, then A is true for $\mathcal{I}$.
7. If x does not occur free in any formula in Γ, then $\Gamma \models \forall x A$ iff $\Gamma \models A$.
8. $\forall x(A \vee B) \models (\forall x A) \vee (\exists x B)$.
9. $\forall x(A \to B), \exists x(A \wedge C) \models \exists x(B \wedge C)$.
10. $\exists x(Fx \to \forall y Fy)$ is true for every interpretation.
11. Not every satisfiable formula has a model.
12. Not every formula that has a model has one in which the domain is finite.

2 Deduction and proof

The semantic definition of logical entailment is fine as an account of what validity means in the abstract, but it does not provide a calculus for establishing good arguments and it does not directly connect with the process of reasoning. An alternative approach is needed, in which the focus will be on inference rather than on truth. We begin with the notion of a derivation.

A derivation of a formula B from a set of formulae Γ is, at its simplest, a finite sequence of formulae $A_1, \ldots, A_n$ such that:

1. The last formula in the sequence, A_n, is B;
2. Every formula A_i in the sequence is either
   (a) a member of Γ, or
   (b) an axiom of logic, or
   (c) an immediate consequence of some formulae in $\{A_1, \ldots, A_{i-1}\}$.

In order to make sense of this definition, we need two further ones: we need to know what are the "axioms of logic" and what is "immediate consequence".

As axioms of first order logic, we take all instances of the following formula schemata. For all formulae A, B, C, and all variables x and terms t:

1. $A \to (B \to A)$
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
3. $(\neg A \to \neg B) \to (B \to A)$
4. $\forall x A \to A^t_x$ where t is free for x in A
5. $A \to \forall x A$ where x is not free in A
6. $\forall x(A \to B) \to (\forall x A \to \forall x B)$
7. $\forall x A$ where A is an axiom of logic

We take the quantifier $\forall x$ to bind more tightly (have smaller scope) than the connective $\to$, so for instance axiom 4 should be read $(\forall x A) \to A^t_x$. By A^t_x we mean the result of substituting term t for all free occurrences of the variable x in the formula A. By saying "t is free for x in A" we mean that x does not occur free in A inside the scope of a quantifier binding any variable that occurs in t. The idea is that by substituting t for x we should not create any bindings that were not there before.

The above axioms use only implication, negation and the universal quantifier. We take all of the other connectives and the existential quantifier to be given by definition, so $A \vee B$ is short for $\neg A \to B$, for instance, and $\exists x A$ is just short for $\neg \forall x \neg A$.

The definition of immediate consequence is extremely simple: B is an immediate consequence of $A \to B$ and A. This is the rule known as detachment, or *modus ponens*.

2.1 The deduction theorem

The axiomatic presentation of logic, as what is known as a *Hilbert system* or a *Frege system*,[2] is quite useful for establishing results *about* logic, because it is easy to run inductions on the length of derivations, but it is hard to use as a way of discovering proofs in logic itself. To make it somewhat easier to handle, a key move is to establish the following theorem linking the arrow of implication to the turnstile of deducability:

Theorem 1 (Deduction Theorem) *For any set of formulae* Γ *and formulae* A *and* B

$$\Gamma \vdash A \to B \quad \textit{iff} \quad \Gamma, A \vdash B$$

Proof: Left to right: trivial. Right to left, induction on the length of the shortest derivation of B from Γ, A. For derivations of length 1, either B is in Γ or B is an axiom of logic or $B = A$. If B is in Γ or is an axiom of logic, $A \to B$ follows from it and the instance $B \to (A \to B)$ of axiom 1. If $B = A$ then $A \to B$ is just $A \to A$, which follows from any Γ (proof given in the lecture).

For derivations of length greater than 1, B must come from some C and $C \to B$ by a step of immediate consequence, and as an induction hypothesis we may suppose we have $\Gamma \vdash A \to (C \to B)$ and $\Gamma \vdash A \to C$. But then we get $\Gamma \vdash A \to B$ by appeal to the instance $(A \to (C \to B)) \to ((A \to C) \to (A \to B))$ of axiom 2 and two steps of immediate consequence. □

2.2 Exercise

Appealing to the deduction theorem as necessary, show:

1. $\vdash$ is a consequence relation.
2. $A \to B, B \to C \vdash A \to C$.
3. $A, \neg A \vdash B$.
4. $\neg\neg A \vdash A$.
5. $A \vdash \neg\neg A$.
6. $A \to B \vdash \neg B \to \neg A$.
7. $A \to \neg(B \to B) \vdash \neg A$.
8. $A \to B, A \to \neg B \vdash \neg A$.
9. If x does not occur free in Γ and $\Gamma \vdash A$ then $\Gamma \vdash \forall x A$.
10. $\forall x(Fx \to Gx), \forall x(Gx \to Hx) \vdash \forall x(Fx \to Hx)$
11. $\forall x(A \to B) \vdash A \to \forall x B$ if x does not occur free in A.
12. $\forall x(Gx \to Gfx) \vdash Ga \to \exists x Gffx$

[2] After David Hilbert and Gottlob Frege who used this style of logical theory early in the development of modern logic.

2.3 Natural deduction

One of the practically most convenient deductive systems is called **Natural Deduction** (ND). It is based on a set of inference rules which naturally reflect the logical meaning of the propositional connectives and provide a well-structured way of formal reasoning, which closely resembles a good and correct informal argumentation.

ND has no axioms, but several inference rules, including a pair of rules for each logical connective: an **introduction rule,** which produces a conclusion containing that connective as the main one, and an **elimination rule,** in which the connective occurs as the main connective of a premise. The rules are listed on page 7. Note that, since $\neg A \equiv A \to \bot$, the rules for $\neg$ can be regarded as particular cases of the corresponding rules for $\to$. Also, there are two additional rules: $(\bot)$ and (RA) which will be discussed further.

The derivation in ND consists of successive application of the inference rules, using as premisses the initial assumptions or already derived formulae, as well as *additional assumptions* which can be added at any step of the derivation. Some of the rules allow for **cancellation** (or, **discharge**) **of assumptions**, which is indicated by putting them in square brackets. The idea of the additional assumptions is that they only play an auxiliary role in the derivation, and when not needed anymore they are cancelled, but *only* at an application of an appropriate rule which allows such cancellation. Note that the cancellation of an assumption, when the rule allows it, is a *right, but not an obligation*, so an assumption can be re-used several times before being cancelled. However, all assumptions which have not been cancelled during the derivation *must be declared* in the list of assumption from which the conclusion is proved to be a logical consequence. Therefore, if we want to prove that a formula C is a logical consequence from a set of assumptions Γ, then any assumption which is not in Γ must be cancelled during the derivation.

Formally, the definition of a derivation given above in the section on Hilbert systems can be contracted a little since the reference to “axioms of logic” can be removed, but then it must be re-expanded to allow formulae that are cancelled by subsequent rule applications, and of course the notion of “immediate consequence” must be extended to cover all of the natural deduction rules.

It turns out that, with appropriate definitions to take care of connectives and quantifiers that are not primitive operations for the axiom system, and with an extra axiom scheme $\bot \to A$ to provide for $\bot$ as an extra primitive, the same sequents are provable by natural deduction as are derivable in the Hilbert system.

2.4 Exercise

Prove the last assertion above. This is the longest exercise in these notes and requires you to go through many cases, but it is useful for making sure you thoroughly understand the deductive systems.

2.5 ND rules for the propositional connectives

(The vertical dots below indicate derivations.)

Introduction rules: **Elimination rules:**

$$(\wedge I) \quad \frac{A, B}{A \wedge B} \qquad\qquad (\wedge E) \quad \frac{A \wedge B}{A}, \quad \frac{A \wedge B}{B}$$

$$(\vee I) \quad \frac{A}{A \vee B}, \quad \frac{B}{A \vee B} \qquad\qquad (\vee E) \quad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

$$(\to I) \quad \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B} \qquad\qquad (\to E) \quad \frac{A, \quad A \to B}{B}$$

$$(\neg I) \quad \frac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A} \qquad\qquad (\neg E) \quad \frac{A, \quad \neg A}{\bot}$$

$$(\bot) \quad \frac{\bot}{A} \qquad\qquad (RA) \quad \frac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}$$

Here is a brief justification of each of the rules:

($\wedge I$) To prove the truth of a conjunction $A \wedge B$, we have to prove the truth of each of A and B.

($\wedge E$) The truth of a conjunction $A \wedge B$, implies the truth of each of the conjuncts.

($\vee I$) The truth of a disjunction $A \vee B$ follows from the truth of either disjunct.

($\vee E$) If the premise is a disjunction $A \vee B$, we reason *per cases*, i.e. we consider separately each of the two possible cases for that disjunction to be true: *Case 1:* A is true, and *Case 2:* B is true. If we succeed to prove that in each of these cases the conclusion C follows, then we have a proof that C follows from $A \vee B$.

($\to I$) To prove the truth of an implication $A \to B$, we assume (in addition to all other premises) that the antecedent A is true and try to prove that the consequent B is true.

($\to E$) This is the Detachment rule (Modus Ponens) which we have already discussed.

($\neg I$) To prove a negation $\neg A$ we can assume A and show that it leads to a contradiction.

($\neg E$) The falsum follows from any contradiction.

($\bot$) "*Ex falso sequitur quodlibet*": from a false assumption anything can be derived.

(RA) This rule '**Reductio ad absurdum**' formalizes the method of *proof by contradiction*: if A can't fail (the assumption $\neg A$ leads to a contradiction) then A must be true.

2.6 Rules for the quantifiers

Introduction rules:

$$(\forall I)^{*} \qquad \frac{A^c_x}{\forall x A}$$

$$(\exists I)^{**} \qquad \frac{A^t_x}{\exists x A}$$

Elimination rules:

$$(\forall E)^{**} \qquad \frac{\forall x A}{A^t_x}$$

$$(\exists E)^{***} \qquad \frac{\exists x A \qquad \begin{matrix}[A^c_x] \\ \vdots \\ C\end{matrix}}{C}$$

* where c is a constant symbol, not occurring in A, nor in any open assumption used in the derivation of A.

** for any term t free for x in A.

*** where c is a constant symbol, not occurring in A, nor in C or in any open assumption in the derivation of C except for A^c_x.

A brief discussion of these rules:

($\forall I$) : What is true of an *arbitrary* thing is true of everything.

($\exists I$) : To prove an existentially quantified sentence $\exists x A(x)$, try to find an explicit *example* c such that $A(c)$.

($\forall E$) : What is true in general (true of everything) is true of each particular thing, including whatever is denoted by t.

($\exists E$) : An existentially quantified sentence $\exists x A(x)$ tells us that there is such a thing as an A. We then say "let's pick one and call it c." Whatever follows, provided it does not depend on the choice of the name c, follows from the existential premise.

3 Soundness and completeness

Here we will give the relevant formal definitions and will outline a generic proof of soundness and completeness for an arbitrary deductive system **D**, which can be applied to any of those studied here (the Hilbert system, the natural deduction system of the sequent calculus formulation of first order logic). As noted, these deductive systems are all equivalent in any case.

Hereafter, by **theory** we mean any set of formulae in the propositional case, or sentences (formulae with no free variables) in the case of first-order logic.

3.1 Soundness and completeness for propositional calculi

We begin with an outline of the completeness proof for propositional deductive systems, but all that follows applies likewise to first-order logic, unless otherwise specified. Note that the deductive power of **D** is needed to prove most of the claims below.

1. *Soundness and consistency.*

   Definition (Soundness1) A deductive system **D** is **sound** if for every theory Γ and a formula A,

   $$\Gamma \vdash_{\mathbf{D}} A \text{ implies } \Gamma \models A.$$

   Definition (Deductive consistency) A theory Γ is **consistent** in **D** (or just, **D**-consistent) if there is no formula A such that $\Gamma \vdash_{\mathbf{D}} A$ and $\Gamma \vdash_{\mathbf{D}} \neg A$. Otherwise, Γ is **D-inconsistent.**

   Definition (Soundness2) A deductive system **D** is **sound** if every satisfiable theory Γ is **D**-consistent.

   The two definitions of soundness are equivalent (exercise).

   Definition (Completeness1) A deductive system **D** is **complete** if for every theory Γ and a formula A,

   $$\Gamma \models A \text{ implies } \Gamma \vdash_{\mathbf{D}} A.$$

   Definition (Completeness2) A deductive system **D** is **complete** if for every theory Γ, if Γ is consistent then Γ is satisfiable.

   Again, the two definitions of completeness are equivalent (exercise).

   Theorem 2 (Soundness of D) *The deductive system* **D** *is sound.*

   To prove soundness, say for the Hilbert system, we show that all of the axioms are true for every interpretation and that the rule of inference (immediate consequence) preserves satisfaction. This is all routine, if a little tedious. Soundness then follows by an easy induction on the lengths of shortest derivations. Since the Hilbert system is sound and the other deductive systems are equivalent to it, they are sound as well.

   Hereafter, '(in)consistent' will mean **D**-(in)consistent. Note well that consistency in the deductive sense has to do withg derivation systems only: it is not a semantic notion.

2. *Some properties of deductive consequence and consistency.*

   (a) $\Gamma \cup \{B\}$ is consistent iff $\Gamma \nvdash_{\mathbf{D}} \neg B$.

   (b) $\Gamma \vdash_{\mathbf{D}} B$ iff $\Gamma \cup \{\neg B\}$ is inconsistent.

   (c) If $\Gamma \cup \{B\}$ is inconsistent and $\Gamma \cup \{\neg B\}$ is inconsistent then Γ is inconsistent.

3. *Maximal consistent theories.*

   A consistent theory Γ is **maximal** if it cannot be extended to a larger consistent theory.

   Proposition 3 *Every maximal consistent theory is closed under deductive consequence in* $\mathbf{D}$.

   This is an obvious outcome of the admissibility of cut, which holds of course for all of the deductive systems we consider.

4. *Some properties of maximal consistent theories.*

   Lemma 4 *A theory* Γ *is a maximal consistent theory iff it is deductively closed in* $\mathbf{D}$ *and for every formula* A, $\Gamma \vdash_{\mathbf{D}} A$ *or* $\Gamma \vdash_{\mathbf{D}} \neg A$.

   Theorem 5 *For every maximal consistent theory* Γ *and formulae* A, B *the following hold:*

   (a) $\neg A \in \Gamma$ *iff* $A \notin \Gamma$.

   (b) $A \wedge B \in \Gamma$ *iff* $A \in \Gamma$ *and* $B \in \Gamma$.

   (c) $A \vee B \in \Gamma$ *iff* $A \in \Gamma$ *or* $B \in \Gamma$.

   (d) $A \to B \in \Gamma$ *iff* $A \in \Gamma$ *implies* $B \in \Gamma$ *(i.e.* $A \notin \Gamma$ *or* $B \in \Gamma$*).*

5. *Lindenbaum's Lemma.*

   Lemma 6 (Lindenbaum's Lemma) *Every consistent theory* Γ *can be extended to a maximal consistent theory.*

   There are several ways to prove Lindenbaum's lemma. We do it by constructing the maximal consistent theory incrementally, starting with the empty set, considering each formula in turn and adding it to the set if it, together with Γ and the set so far, is consistent. The key observation is that this big set is consistent because any derivation of a contradiction from it, being finite, could only involve finitely many of its members and so would be a derivation from one of its finite subsets—which cannot happen, by construction.

6. *Truth lemma for propositional theories.*

   Given a propositional theory Γ, consider the following truth-assignment:

   $$S_\Gamma(p) := \begin{cases} \mathtt{T}, & \text{if } p \in \Gamma, \\ \mathtt{F}, & \text{otherwise.} \end{cases}$$

   for every propositional variable p.

Lemma 7 (Truth Lemma) *If* Γ *is a maximal propositional consistent theory, then for every formula* A, $S_\Gamma(A) = \mathtt{T}$ *iff* $A \in \Gamma$.

7. *Completeness for propositional deductive systems.*

   Corollary 8 *Every maximal consistent theory is satisfiable.*

   Theorem 9 (Completeness of D) *The deductive system* **D** *is complete.*

3.2 Completeness of first-order logic

Now we will establish the main result of this short course: the completeness of any of the deductive systems for first-order logic introduced here. Again, we denote by **D** any of these deductive systems, and fix it hereafter.

The proof builds on the completeness of the propositional fragment of **D**, but requires extra work because instead of a satisfying valuation, we now have to build a whole structure (model) for our consistent theory. There is one additional problem, though: the maximal consistent theory constructed by Lindenbaum's Lemma may not be 'rich' enough to provide all the information needed for the construction of such model. In particular, it may happen that e.g. a formula $\exists x A$ belongs to the maximal consistent theory, while for every term t in the language, free for x in A, $\neg A^t_x$ is in that theory, so the theory contains no 'witness' of the truth of $\exists x A$. We resolve that problem with a few extra technical lemmas, the proofs of which use the deductive power of **D**.

1. *Conservative language extensions.*

   Lemma 10 (Conservative extensions lemma) *If* Γ *is consistent in a language* $\mathcal{L}$*, then it remains consistent when* $\mathcal{L}$ *is extended by the addition of new function symbols (and, in particular, constants).*

2. *Constants and variables.*

   Lemma 11 (Variables lemma) *Let* Γ *be a consistent set and let* $b_1, \ldots, b_n, \ldots$ *be denumerably many constants not occurring in* Γ. *Let* Δ *be the result of replacing each free occurrence of variable* x_i *in* Γ *by the corresponding constant* b_i. *Then:*

   (a) Γ *is consistent iff* Δ *is consistent.*

   (b) Γ *is satisfiable iff* Δ *is satisfiable.*

3. *Henkin witnesses and Henkin extensions of theories.*

   Lemma 12 (Henkin witnesses lemma) *Let* Γ *be consistent in language* $\mathcal{L}$*, let* $b_1, \ldots, b_n, \ldots$ *be denumerably many constants not in* $\mathcal{L}$ *and let* $\mathcal{L}'$ *be the extension of* $\mathcal{L}$ *obtained by adding those constants. Let* $\exists x_1 A_1, \ldots, \exists x_n A_n, \ldots$ *be an enumeration of the existentially quantified formulae of* $\mathcal{L}'$ *chosen such that each* b_i *does not occur in* $\exists x_1 A_1, \ldots, \exists x_i A_i$. *For each* i*, let* B_i *be the formula* $\exists x_i A_i \rightarrow A_i(b_i/x_i)$. *Then the theory* $H(\Gamma) = \Gamma \cup \{B_1, \ldots, B_n, \ldots\}$ *is consistent in language* $\mathcal{L}'$.

The theory $H(\Gamma)$ defined above clearly has the property that for every formula $\exists x A$ such that $H(\Gamma) \vdash \exists x A$, there is a constant symbol c in the language of Γ^H, such that $H(\Gamma) \vdash A^c_x$. A theory with this property is called a **Henkin theory**.

4. *Lindenbaum's lemma for first-order theories.*

   Lemma 13 (Lindenbaum's Lemma) *Every consistent first-order theory* Γ *can be extended to a maximal consistent theory* Γ^*. *Moreover, if* Γ *is a Henkin theory, then* Γ^* *is a Henkin theory, too.*

5. *Building canonical models from maximal Henkin theories.*

   Given a maximal consistent Henkin theory Δ, we can construct a model for Δ by taking as domain the set of ground terms of its language. Then we interpret that language in a 'canonical' way: every ground term designates itself, and the predicate symbols are interpreted according to what Δ dictates. The resulting structure satisfies Δ.

6. *Completeness proof completed.*

   Now, the completeness proof follows immediately: take a consistent theory Γ; construct a Henkin theory Γ^H extending Γ; then extend Γ^H to a maximal consistent Henkin theory Δ, and construct a model for Δ as above. That model will, in particular, satisfy Γ.

The first six books in the following list are suggestions for elementary logic texts that you might wish to read as background. There is no need to read more than one of them, as they all cover pretty much the basics.

The other books in the list are treatments of mathematical logic which include a lot more metatheory than we have been able to cover in this course. If you intend to go further with mathematical logic, you will need to study one or more of them.

References

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